International Journal of Theoretical Physics, Vol. 45, No. 11, November 2006 (© 2006)
DOI: 10.1007/s10773-006-9184-2

Relativities and Homogeneous Spaces I[1]

Heinrich Saller[2]

Received February 8, 2006; Accepted March 13, 2006
Published Online: May 26, 2006

Special relativity, the symmetry breakdown in the electroweak standard model, and the dichotomy of the spacetime related transformations with the Lorentz group, on the one side, and the chargelike transformations with the hypercharge and isospin group, on the other side, are discussed under the common concept of "relativity." A relativity is defined by classes G/H of "little" group in a "general" group of operations. Relativities are representable as linear transformations that are considered for five physically relevant examples.

KEY WORDS: operation group; induced representations; relativities; homogeneous spaces.

1. FIVE RELATIVITIES FOR AN INTRODUCTION

Basic physical theories involve both external and internal degrees of freedom that are acted on, respectively, by operations from the Poincaré group, i.e., Lorentz group and spacetime translations, and by operations from the hypercharge, isospin and color group. The properties of all basic interactions and particles are determined and characterized by invariants and eigenvalues for these operation groups. Although the product of external and internal operations in the acting group is direct, the internal "chargelike" operations are coupled to the external "spacetime-like" ones: any spacetime translation is accompanied by a chargelike operation. This is implemented by the gauge fields in the standard model of electroweak and strong interactions. In the following, the dichotomy and the connection of external and internal operations will be discussed under the label "unitary relativity," especially with respect to its representations by interactions and particles.

To see its general and its specific structures, unitary relativity will be introduced and considered as one example in five relativities: perpendicular relativity as realized after discovering the surface of the earth to be spherical, ro-

[1] Finite Dimensional Relativity Representations.

[2] Max-Planck-Institut für Physik, Werner-Heisenberg-Institut, München, Germany; e-mail: hns@mppmu.mpg.de

tation relativity, or space and time relativity, as used in what we call special relativity with "timelike" and "spacelike" translations, Lorentz group relativity, or Minkowski spacetime relativity, as an important ingredient of general relativity, electromagnetic relativity as formulated in the standard model of electroweak interactions Weinberg (1967) and finally, and that is mostly new, unitary relativity.

Relativity will be defined by operation groups, an example: In special relativity, the distinction of your rest system determines a decomposition of spacetime translations into time and position translations. Compatible with this decomposition is your position rotation group $\mathbf{SO}(3)$ as a subgroup of the orthochronous Lorentz group $\mathbf{SO}_0(1, 3)$. There are as many decompositions of spacetime into time and position as there are rotation groups in a Lorentz group. The rotation group classes are parametrizable by the points of a one shell 3-dimensional hyperboloid $\mathcal{Y}^3 \cong \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$ that give the momenta (velocities) for all possible motions. Another example: The perpendicularities of mankind, if earthbound, are characterized by the axial rotation groups in a rotation group and parametrizable by coordinates of the earth's surface $\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$.

Now in general: The choice of an "idolized" operation group H in a "general" operation group G picks one element in the G-symmetric space G/H, which stands for the relativity of the "idolized" group, called H-relativity. An "idolization" Bacon (1994) goes, negatively, with the "narrow-minded" assumption of an absolute point of view or, positively, with the distinction of a smaller operation symmetry, enforced, e.g., by initial or boundary conditions. Important examples are degenerate ground states ("spontaneous symmetry breakdown") where an "interaction-symmetry" G is reduced to a "particle-symmetry" H, e.g., the degenerate ground states of superconductivity, of superfluidity, of a ferromagnetic or of the electroweak standard model. The ground state degeneracy is characterized by the symmetric space G/H.

This gives the first four columns of the following table, which together with the last one will be discussed with their representations in more detail below

relativity	"general" group $G(r, r_{\mathcal{R}})$	"idolized" subgroup H	homogeneous space G/H	relativity parameters
axial rotation	$\mathbf{SO}(3)$	$\mathbf{SO}(2)$	2-Sphere	2 Transversal
(perpendicular)	$\sim\mathbf{SU}(2)$		$\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$	coordinates
relativity	$(1, 0)$		$\cong \mathbf{SU}(2)/\mathbf{SO}(2)$	
Rotation	$\mathbf{SO}_0(1, 3)$	$\mathbf{SO}(3)$	3-Hyperboloid	3 Momenta
(special)	$\sim\mathbf{SL}(\mathbb{C}^2)$	$\sim\mathbf{SU}(2)$	$\mathcal{Y}^3 \cong \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$	
relativity	$(2, 1)$		$\cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$	
Lorentz group	$\mathbf{GL}(\mathbb{R}^4)$	$\mathbf{O}(1, 3)$	tetrad or metric manifold	10 Components
(general)	$(4, 4)$		$\mathcal{M}^{10} \cong \mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3)$	for metric tensor
relativity			$\cong \mathbf{D}(1) \times \mathbf{SO}_0(3, 3)/\mathbf{SO}_0(1, 3)$	

relativity	"general" group $G(r, r_{\mathcal{R}})$	"idolized" subgroup H	homogeneous space G/H	relativity parameters
Electro-magnetic relativity	$\mathbf{U}(2)$ $(2, 0)$	$\mathbf{U}(1)_+$	Goldstone manifold $\mathcal{G}^3 \cong \mathbf{U}(2)/\mathbf{U}(1)_+$	3 Weak coordinates
Unitary relativity	$\mathbf{GL}(\mathbb{C}^2)$ $(4, 2)$	$\mathbf{U}(2)$	Positive 4-cone $\mathcal{D}^4 \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$ $\cong \mathbf{D}(1) \times \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$	4 Spacetime coordinates

orientation manifolds of five relativities

Somewhat in accordance with the historical development, the "general" operations of one relativity can constitute the "idolized" group of the next relativity as seen in the two chains ending in full general linear groups, a real one for spacetime concepts, from flat to spherical earth to special and general relativity, and a complex one for interactions, from electromagnetic to electroweak transformations and their spacetime (gauge) dependence:

$$\begin{aligned} &\mathbf{SO}(2) \quad \subset \mathbf{SO}(3) \quad \subset \mathbf{SO}_0(1, 3) \quad \subset \mathbf{GL}(\mathbb{R}^4),\\ &\mathbf{U}(1)_+ \quad \subset \mathbf{U}(2) \quad \subset \mathbf{GL}(\mathbb{C}^2). \end{aligned}$$

All groups in the five relativities considered are real Lie groups. All "general" groups are reductive, for perpendicular and rotation relativity even semisimple. Perpendicular and electromagnetic relativity have a compact "general" group. With the exception of Lorentz group relativity, all "idolized" groups are compact subgroups. The 2nd column contains the dimension of the maximal abelian subgroups, which is the rank r of the group G generating Lie algebra $L = \log G$, and of the maximal noncompact abelian subgroups, i.e., the real rank $r_{\mathcal{R}}$. With the exception of $\mathbf{GL}(\mathbb{R}^4)$, the maximal abelian subgroups allow a unique decomposition into compact Cartan torus and noncompact Cartan plane. A Cartan torus is a direct product of circle groups in the form $\mathbf{U}(1) = \exp i\mathbb{R}$ or $\mathbf{SO}(2) \cong \exp \sigma_3 i\mathbb{R}$, a Cartan plane is a direct product of additive line groups $\mathbb{R}$ (translations), which also can be used in the multiplicative form $\mathbf{D}(1) = \exp \mathbb{R}$ or $\mathbf{SO}_0(1, 1) \cong \exp \sigma_3\mathbb{R}$. The rank gives the number of independent invariants, rational or continuous $r = n_{\mathcal{I}} + n_{\mathcal{R}}$, that characterize a G-representation. The real rank is the maximal number of the continuous invariants $n_{\mathcal{R}} \leq r_{\mathcal{R}}$.

Unitary relativity $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \cong \mathbf{D}(1) \times \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$, i.e., the complex linear relativization of the maximal compact subgroup with the internal "chargelike" hypercharge and isospin operations $\mathbf{U}(2)$, is parametrized by a noncompact real 4-dimensional homogeneous space, called causal spacetime $\mathcal{D}^4$, a name to be justified below. Unitary relativity is visible in the spacetime dependence of quantum fields, which represent the internal operations. The representations of unitary relativity $\mathcal{D}^4$ with a 2-dimensional Cartan plane are characterized by two continuous invariants, which, in appropriate units, can be taken as two masses. The $\mathcal{D}^4$-representations determine the spacetime interactions with their normalization, especially the gauge interactions with their coupling constants related to the ratio of the two invariants, and, for the $\mathcal{D}^4$-tangent translations $\mathbb{R}^4$, the particles and their masses. The common language for interactions and elementary particles is the representation theory and harmonic analysis of unitary relativity (more below).

There is a mathematical framework, almost tailored for relativities: the theory of induced representations, pioneered by Frobenius (1968), used for free particles by Wigner (1939) and worked out for noncompact groups especially by Mackey (1951). There, a subgroup H-representation induces a full group G-representation leading to a $G \times H$-representation as subrepresentation of the two-sided regular $G \times G$-representation. Such a dichotomic transformation property with a doubled group, $G \times G$ as group and "isogroup," is familiar, with respect to the Lorentz and the isospin group, $\mathbf{SU}(2) \times \mathbf{SU}(2)$ as spin and isospin, from the fields in the electroweak standard model. Especially for noncompact nonabelian groups, the theory is not easy to penetrate. All the mathematical details are given in the textbooks of Helgason (1984), Knapp (1986) and Folland (1995) and, especially for distributions, of Treves (1967).

In the following, only some motivating and qualititive mathematical remarks will be given with respect to this theory, which will be used in physical implementations. The first part of this paper works with finite-dimensional relativity structures, which may be not so familiar in such a conceptual framework. After a parametrization of the relativity manifold G/H, there will be given its fundamental representations, called transmutators, which mediate the transition from an idolized group H to the full group G. With the fundamental transmutators all finite dimensional relativity representations can be constructed as used, e.g., in the transition from the fields for the electroweak interactions to the asymptotic particles.

The mathematically more demanding second part ("Spacetime and unitary relativity") uses representations of noncompact operation groups on Hilbert spaces, necessarily infinite-dimensional for faithful representations.

2. RELATIVITY PARAMETERS

There are operation-induced parameters for the real homogeneous relativity spaces G/H, e.g., the three momenta (velocities) for rotation (special) relativity or the three weak coordinates of electromagnetic relativity as used in the mass modes of the three weak bosons.

The action of a "general" group G on a set S, denoted by $\bullet$, decomposes S into disjoint orbits $G \bullet x$ for $x \in S$ that are isomorphic to subgroup classes $G \bullet x \cong G/H$ where the "idolized" group H is the fixgroup (fixer, "little" group, isotropy group) G_x of the G-action. The elements of homogeneous spaces $gH \in G/H$ are group subsets (cosets), e.g., position rotation groups in a Lorentz group or electromagnetic transformation groups in a hypercharge-isospin group. The cosets have representatives $g_r \in gH \in G/H$, written as $g_r \Subset G/H$, which can be characterized by what will be called "relativity parameters," a real parametrization of the subgroup classes.

Relativity parameters can be obtained via orbit parametrizations. The real Lie groups considered are linear groups $H \subseteq G \subseteq \mathbf{GL}(V)$, acting on real or complex vector spaces V. The orbit $G \bullet x$ parametrizes the homogeneous space G/H by V-vectors and their components with respect to a basis,

$$\begin{aligned} &x \in V,\ H \cong G_x = \{g \in G \mid g \bullet x = x\} \\ &\Rightarrow G/H \cong G \bullet x \subseteq V. \end{aligned}$$

2.1. Weak Coordinates for Electromagnetic Relativity

The hypercharge-isospin group $\mathbf{U}(2)$ acts, in the defining representation, on a complex 2-dimensional vector space:

$$\mathbf{U}(2) \ni u = e^{i\alpha_0} \begin{pmatrix} e^{i\alpha_3} \cos\frac{\theta}{2} & -e^{-i\varphi} \sin\frac{\theta}{2} \\ e^{i\varphi} \sin\frac{\theta}{2} & e^{-i\alpha_3} \cos\frac{\theta}{2} \end{pmatrix}.$$

Each nontrivial vector has a $\mathbf{U}(1)$-isomorphic fixgroup, e.g. e^2, which defines $\mathbf{U}(1)_+$ as an "idolized" electromagnetic subgroup,

$$\mathbb{C}^2 \cong V \ni e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} e^{2i\alpha_0} & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{U}(2)_{e^2} = \mathbf{U}(1)_+.$$

The orbits of the chosen vector, here $u \bullet e^2$, and its $\mathbf{U}(2)$-orthonormal partner, here $(u \bullet e^2)_\perp$, give the two columns of the matrix parametrization $v \in \mathbf{U}(2)$ of the Goldstone manifold $\mathcal{G}^3$,

$$\begin{aligned} &\mathcal{G}^3 \cong \mathbf{U}(2)/\mathbf{U}(1)_+ \cong \{((u \bullet e^2)_\perp, u \bullet e^2) = v \mid u \in \mathbf{U}(2)\}, \\ &v = \begin{pmatrix} e^{i(\alpha_3-\alpha_0)} \cos\frac{\theta}{2} & -e^{-i(\varphi-\alpha_0)} \sin\frac{\theta}{2} \\ e^{i(\varphi-\alpha_0)} \sin\frac{\theta}{2} & e^{-i(\alpha_3-\alpha_0)} \cos\frac{\theta}{2} \end{pmatrix}. \end{aligned}$$

In the standard model of electroweak interactions the vector space $V \cong \mathbb{C}^2$ desribes the chargelike degrees of freedom of the Higgs field. The three weak parameters $(\alpha_3 - \alpha_0, \varphi - \alpha_0, \theta)$ parametrize electromagnetic relativity. As manifold, not as group, the Goldstone manifold $\mathcal{G}^3$ is isomorphic to $\mathbf{SU}(2)$.

2.2. Orbits of Metric Tensors

With the exception of electromagnetic relativity, all relativity parameters will by given by the "general" group G-orbit of a metric invariant under the action of an "idolized" subgroup H. In this context, the homogeneous space G/H for H-relativity was called, by Weyl (1923), orientation manifold of the metric (bilinear or sesquilinear product).

The invariance of a metric[3] γ with respect to the action of a linear group,

$$\mathbf{GL}(V) \supset H \ni g, \quad \gamma(x, y) \longmapsto \gamma(g \bullet x, g \bullet y) = \gamma(x, y) \text{ for all } x, y \in V,$$

gives the parametrization of the fixgroup classes by the orbit of the metric tensor γ,

$$\{g \in G \subseteq \mathbf{GL}(V) \,\big|\, g \circ \gamma \circ g^* = \gamma\} = H \Rightarrow \{g \circ \gamma \circ g^* \,\big|\, g \in G\} \cong G/H.$$

2.3. Metric Tensor for Lorentz Group Relativity

A bilinear form (metric) of a vector space V is a power two tensor $\gamma \in V^T \otimes V^T$ with the dual vector space V^T (linear forms). For an n-dimensional space, the subspaces $V^T \wedge V^T$, totally antisymmetric, denoted by $\wedge$, and $V^T \vee V^T$, totally symmetric, denoted by $\vee$, have the dimensions $\binom{n}{2}$ and $\binom{n+1}{2}$ respectively.

A real vector space $V \cong \mathbb{R}^n$ has a causality structure by embedding the cone of the positive numbers $\mathbb{R}_+ \longrightarrow V_+ \subset V$ into the "future" cone of the vector space $x \succeq 0 \iff x \in V_+$. A nontrivial "future" cone $V_+ \neq \{0\}$ can be defined by a bilinear symmetric form with "causal" signature $(t, s) = (1, s)$ invariant under the generalized Lorentz group $\mathbf{SO}_0(1, s)$. Such a causality structure for $V \cong \mathbb{R}^{1+s}$ is familiar for time $\mathbb{R}$ with total order and Minkowski spacetime $\mathbb{R}^4$ with the special relativistic partial order.

Any metric tensor of $V \cong \mathbb{R}^4$ with causal signature $(1, 3)$, e.g., an orthonormal Lorentz metric tensor,

$$V \cong \mathbb{R}^4, \quad \eta = \begin{pmatrix} 1 & 0 \\ 0 & -\mathbf{1}_3 \end{pmatrix} \in V^T \vee V^T,$$

defines an "idolized" Lorentz group as invariance group. Its $\mathbf{GL}(\mathbb{R}^4)$-orbit leads to a parametrization of the metric manifold with dimension $\binom{5}{2} = 10$ for Lorentz group relativity

$$\mathcal{M}^{10} \cong \mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1, 3) \cong \{h \circ \eta \circ h^T = \gamma \,\big|\, h \in \mathbf{GL}(\mathbb{R}^4)\}, \tag{1}$$

$$\gamma \cong \gamma^{\mu\nu} = h^\mu_j \eta^{jk} h^\nu_j = \gamma^{\nu\mu}; \tag{2}$$

here $\mu, j \in \{0, 1, 2, 3\}$.

[3] Usual notation for the metric $\gamma^{\mu\nu} = g^{\mu\nu}$, here $g \in G$ is reserved for group elements.

2.4. Spherical Coordinates for Perpendicular Relativity

With the local isomorphy of the rotation group to the spin group $\mathbf{SO}(3) \sim \mathbf{SU}(2)$ an "idolized" axial rotation subgroup $\mathbf{SO}(2) \subset \mathbf{SU}(2)$ is given by the invariance group of the hermitian and traceless Pauli matrix σ_3. Its $\mathbf{SU}(2)$-orbit leads to the 2-sphere parametrization of perpendicular relativity,

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2) \cong \left\{ u \circ \sigma_3 \circ u^\star = \frac{\vec{x}}{r} \,\Big|\, u \in \mathbf{SU}(2) \right\},$$
$$\frac{\vec{x}}{r} \cong \frac{\vec{x}^\alpha_\beta}{r} = u^\alpha_j \sigma_3{}^j_k u^{\star k}{}_\beta;$$

here $\alpha, j \in \{1, 2\}$. The two angles (spherical coordinates) in the traceless hermitian matrix $\frac{\vec{x}}{r}$ can be parametrized by three position translations with one condition for the determinant,

$$\frac{\vec{x}}{r} = \frac{\vec{x}^\star}{r} = \begin{pmatrix} \cos\theta & e^{-i\varphi}\sin\theta \\ e^{i\varphi}\sin\theta & -\cos\theta \end{pmatrix} = \frac{1}{r}\begin{pmatrix} x_3 & x_1 - ix_2 \\ x_1 + ix_2 & -x_3 \end{pmatrix}$$

with $\operatorname{tr} \frac{\vec{x}}{r} = 0$ and $-\ \det \frac{\vec{x}}{r} = \frac{\vec{x}^2}{r^2} = 1$.

The restriction uses the rotation $\mathbf{SO}(3)$-invariant product $\vec{x}^2 = x_3^2 + x_1^2 + x_2^2$ in three dimensions.

2.5. Momenta for Rotation Relativity

An "idolized" rotation group $\mathbf{SO}(3)$ in a Lorentz group $\mathbf{SO}_0(1, 3)$ is characterized by a distinguished definite metric of a real 3-dimensional vector space (position), e.g., $\gamma = \mathbf{1}_3$. Similarly, one can work with a sesquilinear scalar product δ of a complex 2-dimensional space $V \cong \mathbb{C}^2$ invariant under the locally isomorphic spin group $\mathbf{SU}(2) \sim \mathbf{SO}(3)$ in the special linear group $\mathbf{SL}(\mathbb{C}^2) \sim \mathbf{SO}_0(1, 3)$. The $\mathbf{SL}(\mathbb{C}^2)$-orbit of the metric parametrizes rotation relativity by the points of an energy-momentum 3-hyperboloid,

$$\delta = \mathbf{1}_2, \quad \mathcal{Y}^3 \cong \mathbf{SO}_0(1, 3)/\mathbf{SO}(3) \cong \left\{ s \circ \delta \circ s^\star = \frac{q}{m} \,\Big|\, s \in \mathbf{SL}(\mathbb{C}^2) \right\},$$
$$\frac{q}{m} \cong \frac{q^A_{\dot{B}}}{m} = s^A_\alpha \delta^\alpha_\beta s^{\star\beta}{}_{\dot{B}};$$

here $A, \alpha \in \{1, 2\}$. The three real hyperbolic coordinates in the hermitian matrix $\frac{q}{m}$ can be chosen from four energy-momenta with one condition for the determinant,

$$\frac{q}{m} = \frac{q^\star}{m} = \begin{pmatrix} \cosh 2\beta + \cos\theta \sinh 2\beta & e^{-i\varphi} \sin\theta \sinh 2\beta \\ e^{i\varphi} \sin\theta \sinh 2\beta & \cosh 2\beta - \cos\theta \sinh 2\beta \end{pmatrix}$$

$$= \frac{1}{m} \begin{pmatrix} q_0 + q_3 & q_1 - iq_2 \\ q_1 + iq_2 & q_0 - q_3 \end{pmatrix}$$

with $\det \frac{q}{m} = \frac{q^2}{m^2} = 1$.

The restriction of the four energy-momenta to the three momenta uses the $\mathbf{SO}_0(1, 3)$-invariant bilinear form $q^2 = q_0^2 - \vec{q}^{\,2}$.

2.6. Spacetime Future for Unitary Relativity

An "idolized" unitary group $\mathbf{U}(2)$, called hyperisospin group, a maximal compact subgroup of the general linear group $\mathbf{GL}(\mathbb{C}^2)$, called extended Lorentz group, is given by the invariance group of a scalar product δ of a complex 2-dimensional vector space. The $\mathbf{GL}(\mathbb{C}^2)$-orbit defines four real parameters for unitary relativity, i.e., for the orientation manifold of the $\mathbf{U}(2)$-scalar product,

$$\delta = \mathbf{1}_2, \quad \mathcal{D}^4 \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \cong \{\psi \circ \delta \circ \psi^\star = x \,|\, \psi \in \mathbf{GL}(\mathbb{C}^2)\},$$

$$x \cong x^A_{\dot{B}} = \psi^A_\alpha \delta^\alpha_\beta \psi^{\star\,\beta}_{\dot{B}},$$

$$x = x^\star = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix};$$

here $A, \alpha \in \{1, 2\}$. These four real orbit parameters characterize the strictly positive elements in the C^*-algebra of complex (2×2) matrices,

$$x = \psi \circ \psi^\star \iff x = x^\star \text{ and } \operatorname{spec} x > 0$$

$$\iff \det x = x^2 > 0 \text{ and } \operatorname{tr} x = 2x_0 > 0.$$

They describe the absolute modulus set in the polar decomposition of $\mathbf{GL}(\mathbb{C}^2)$ into noncompact classes for the maximal compact group with the unitary phases,

$$\mathbf{GL}(\mathbb{C}^2) \ni \psi = |\psi| \circ u \in \mathbf{D}(2) \circ \mathbf{U}(2),$$

$$\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \cong \mathbf{D}(2) \ni |\psi| = \sqrt{\psi \circ \psi^\star} = \sqrt{x}.$$

The positive matrices x are parametrizable by the points of the open future cone in flat Minkowski spacetime,

$$\mathcal{D}^4 \cong \mathbb{R}^4_+ = \{x \in \mathbb{R}^4 \,\big|\, x^2 > 0, x_0 > 0\}.$$

The cone manifold is embeddable into its own tangent space, the spacetime translations $\mathbb{R}^4 \supset \mathcal{D}^4$. They inherit the action of the dilation extended orthochronous Lorentz group $\mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(1) \cong \mathbf{D}(1) \times \mathbf{SO}_0(1,3)$, which constitutes the homogeneous part in the extended Poincaré group $[\mathbf{D}(1) \times \mathbf{SO}_0(1,3)] \vec{\times} \mathbb{R}^4$.

3. RELATIVITY TRANSITIONS

Elements of a relativity, i.e., of a homogeneous space G/H, are related to each other by the action of the full group G, e.g., different perpendicularities by rotations of the earth's surface or different nonrelativistic space-times by Lorentz transformations of spacetime.

With real parameters for H-relativity G/H one can partly parametrize the "general" group G. Each coset can be given a defining representative $g_r \in gH \subseteq G$ as linear operation on a complex vector space. Such representatives have a characteristic two-sided $G \times H$ transformation behavior in the group $G \times G$, called relativity transition or transmutation from the "general" group to the "idolized" group: A left multiplication of the representative $g_r \in gH$ by $k \in G$ hits the chosen representative $(kg)_r \in kgH$ up to a right multiplication with an H-element,

$$k \in G, \quad kg_r = (kg)_r h(g_r, k) \text{ with } h(g_r, k) \in H.$$

The group action $k \in G$ is accompanied by an action from the "idolized" subgroup $h(g_r, k) \in H$, which depends on the representative g_r. It is called Wigner element and Wigner subgroup-operation, in generalization of the familiar Wigner rotation, which arises from a Lorentz transformation of a boost.

3.1. From Interaction Group to Particle Group

An example where both electromagnetic relativity with the transition $\mathbf{U}(2) \rightarrow \mathbf{U}(1)_+$ and rotation (special) relativity with the transition $\mathbf{SL}(\mathbb{C}^2) \rightarrow \mathbf{SU}(2)$ play a role is the transition from relativistic electroweak interaction fields to particles in the standard model Saller (2001),

$$\begin{array}{ccccc} \mathbf{SL}(\mathbb{C}^2) \times & \mathbf{U}(2) & \longrightarrow \mathbf{SU}(2) \times & \mathbf{U}(1)_+ \\ \text{Lorentz} & \text{hypercharge} - \text{isospin} & \text{spin} & \text{electromagnetic} \end{array}$$

For example, the lepton field in the minimal model connects, for each spacetime translation, the two $\mathbf{SL}(\mathbb{C}^2)$-degrees of freedom with the two isospin $\mathbf{SU}(2)$ degrees of freedom and a hypercharge $\mathbf{U}(1)$ value $y = -\frac{1}{2}$

$$\mathbb{R}^4 \ni x \longmapsto \mathbf{l}(x)^A_\alpha \text{ with } A, \alpha \in \{1, 2\}.$$

The transition from interaction field to particles with respect to internal degrees of freedom uses the ground state degeneracy, implemented by the

$\mathbf{U}(2)$-invariant condition $\langle \Phi^\star \Phi(x) \rangle = M^2 > 0$ with the Lorentz scalar Higgs field

$$\mathbb{R}^4 \ni x \longmapsto \Phi(x)^\alpha.$$

It is an isospin doublet with hypercharge $y = \frac{1}{2}$. The Higgs field transmutes from "general" hyperisospin $\mathbf{U}(2)$-properties of the lepton field to "idolized" electromagnetic $\mathbf{U}(1)_+$-properties of the particles, e.g., for the electron-positron field, an isosinglet with electromagnetic charge number $z = -1$,

$$\mathbf{U}(2) \longrightarrow \mathbf{U}(1)_+ : \ \ \mathbf{l}(x)^A_\alpha \longmapsto \mathbf{e}(x)^A = \frac{\Phi(x)^\alpha}{|\Phi|(x)} \mathbf{l}(x)^A_\alpha = \mathbf{l}(x)^A_2 + \ldots$$

The "idolization" comes with the distinction of a ground state and the expansion of the Higgs transmutator (more below) $\frac{\Phi^\alpha(x)}{|\Phi|(x)} = \delta^\alpha_2 + \ldots$ for $e^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cong \delta^\alpha_2$ and $|\Phi|(x) = \sqrt{\Phi^\star \Phi(x)}$.

With respect to external degrees of freedom, the transition from a left-handed Weyl field with Lorentz group $\mathbf{SL}(\mathbb{C}^2)$-action to particles with mass $m > 0$ and $\mathbf{SU}(2)$-spin requires a rest system. The related harmonic expansion of the spacetime field with respect to eigenvectors involves the electron creation and positron annihilation operators $\mathrm{u}^a(\vec{q})$ and $\mathrm{a}^{\star a}(\vec{q})$ respectively for spin directions $a \in \{1, 2\}$ and momentum $\vec{q}$ as translation eigenvalues,

$$\begin{aligned} \mathbf{SL}(\mathbb{C}^2) \longrightarrow \mathbf{SU}(2) : \quad & e(x)^A \longmapsto \mathrm{u}(\vec{q})^a, \ \mathrm{a}^\star(\vec{q})^a \\ \text{where } e(x)^A = & {}^{\oplus}\!\!\int \frac{d^3 q}{2q_0} s\left(\frac{q}{m}\right)^A_a [e^{iqx} \mathrm{u}(\vec{q})^a + e^{-iqx} \mathrm{a}^\star(\vec{q})^a] \\ \text{with } q_0 = & \sqrt{m^2 + \vec{q}^2}. \end{aligned}$$

The boost representation $s(\frac{q}{m})^A_a$, discussed below as Weyl transmutator, connects the Lorentz group $\mathbf{SL}(\mathbb{C}^2)$-action for fields with a rest system spin $\mathbf{SU}(2)$-action for massive particles.

Altogether, there are four transmutators involved with $G \times H$-transformations for four different group pairs $H \subset G$: the lepton field with external-internal transformation behavior, the Higgs field as internal transmutator from interaction to particles, the boost representation as corresponding external transmutator, which leaves the creation and annihilation operators with the external-internal properties of the particles (spin and charge)

$$\begin{array}{ccc} & \text{interactions} & \\ & \mathbf{l}(x)^A_\alpha & \\ \mathbf{GL}(\mathbb{C}^2) & \longleftrightarrow & \mathbf{U}(2) \\ \text{external } s\left(\frac{q}{m}\right)^a_A \updownarrow & & \updownarrow \Phi(x)^\alpha \text{ internal} \\ \mathbf{SU}(2) & \longleftrightarrow & \mathbf{U}(1)_+ \\ & \mathrm{u}(\vec{q})^a,\ \mathrm{a}^\star(\vec{q})^a & \\ & \text{particles} & \end{array}$$

3.2. Pauli Transmutator

Perpendicular relativity, parametrizable by a 2-sphere of radius r, is represented as linear operator by the fundamental Pauli transmutator from rotations to axial rotations,

$$\begin{aligned} \mathbb{R}^3 \supset \Omega^2 \ni \frac{\vec{x}}{r} \longmapsto u\left(\frac{\vec{x}}{r}\right) \in \mathbf{SU}(2),& \\ u\left(\frac{\vec{x}}{r}\right) \circ \sigma_3 \circ u^\star \left(\frac{\vec{x}}{r}\right) &= \frac{\vec{x}}{r} = \frac{\sigma_a x_a}{r} \text{ with } r^2 = \vec{x}^2, \\ u\left(\frac{\vec{x}}{r}\right) = e^{i\vec{\alpha}} &= \mathbf{1}_2 \cos\alpha + i\frac{\vec{\alpha}}{\alpha}\sin\alpha \text{ with } \tan 2\alpha = \tan\theta = \frac{\sqrt{x_1^2 + x_2^2}}{x_3} \\ &= \sqrt{\frac{x_3 + r}{2r}}\left[\mathbf{1}_2 + i\frac{\vec{x}_\perp}{x_3 + r}\right] \\ &= \frac{1}{\sqrt{2r(x_3 + r)}}\begin{pmatrix} x_3 + r & -x_1 + ix_2 \\ x_1 + ix_2 & x_3 + r \end{pmatrix} \\ = u(\varphi, \theta) &= \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \end{aligned}$$

An action on the Pauli transmutator $u(\frac{\vec{x}}{r})$ from left with the spin group $\mathbf{SU}(2)$ gives the transmutator at the rotated point $O.\vec{x}$ on the 2-sphere and a right action with the axial group $\mathbf{SO}(2)$ (Wigner axial rotation)

$$o \in \mathbf{SU}(2):\ o \circ u\left(\frac{\vec{x}}{r}\right) = u\left(\frac{O.\vec{x}}{r}\right) \circ v\left(o, \frac{\vec{x}}{r}\right)$$

$$\text{with } \begin{cases} v\left(o, \frac{\vec{x}}{r}\right) \in \mathbf{SO}(2), \\ O.\vec{x} = o \circ \vec{x} \circ o^\star, \\ O^b_a = \frac{1}{2}\operatorname{tr}\sigma_a \circ o \circ \sigma_b \circ o^\star \in \mathbf{SO}(3). \end{cases}$$

The explicit complicated looking expression for the Wigner axial rotation can be computed from $v(o, \frac{\vec{x}}{r}) = u^\star(\frac{O.\vec{x}}{r}) \circ o \circ u(\frac{\vec{x}}{r})$.

3.3. Weyl Transmutators

In special relativity, the Weyl representations of the boosts, parametrized by the energy-momentum hyperboloid for mass $m > 0$, are a familiar example for a transmutator,

$$\mathbb{R}^4 \supset \mathcal{Y}^3 \ni \frac{q}{m} \longmapsto s\left(\frac{q}{m}\right) \in \mathbf{SL}(\mathbb{C}^2),$$
$$s\left(\frac{q}{m}\right) \circ \mathbf{1}_2 \circ s^\star\left(\frac{q}{m}\right) = \frac{q}{m} = \frac{\sigma^j q_j}{m} \text{ with } m^2 = q^2$$

and Weyl matrices $\sigma^j = (\mathbf{1}_2, \vec{\sigma})$ and $\check{\sigma}^j = (\mathbf{1}_2, -\vec{\sigma})$. The explicit expressions involve the Pauli transmutator for the two spherical degrees of freedom:

$$\frac{q}{m} = u\left(\frac{\vec{q}}{|\vec{q}|}\right) \circ e^{2\beta\sigma_3} \circ u^\star\left(\frac{\vec{q}}{|\vec{q}|}\right),$$
$$e^{2\beta\sigma_3} = \text{diag}\,\frac{q}{m} = \frac{1}{m}\begin{pmatrix} q_0 + |\vec{q}| & 0 \\ 0 & q_0 - |\vec{q}| \end{pmatrix}, \quad \tanh 2\beta = \frac{|\vec{q}|}{q_0} = \frac{v}{c},$$
$$s\left(\frac{q}{m}\right) = u\left(\frac{\vec{q}}{|\vec{q}|}\right) \circ e^{\beta\sigma_3} = \mathbf{1}_2 \cosh\beta + \frac{\vec{q}}{|\vec{q}|}\sinh\beta$$
$$= \sqrt{\frac{q_0 + m}{2m}}\left[\mathbf{1}_2 + \frac{\vec{q}}{q_0 + m}\right]$$
$$= \frac{1}{\sqrt{2m(q_0 + m)}}\begin{pmatrix} q_0 + q_3 + m & -q_1 + iq_2 \\ q_1 + iq_2 & q_0 - q_3 + m \end{pmatrix}.$$

The left-handed Weyl transmutator $s(\frac{q}{m}) \in \mathbf{SL}(\mathbb{C}^2)$ together with its right-handed partner $\hat{s}(\frac{q}{m}) = u(\frac{\vec{q}}{|\vec{q}|}) \circ e^{-\beta\sigma_3} \in \mathbf{SL}(\mathbb{C}^2)$ where $\hat{s} = s^{-1\star}$ are the two fundamental transmutators from Lorentz group to rotation subgroups. The restriction in the energy-momenta from four to three parameters by the on-shell hyperboloid $\mathcal{Y}^3$ condition $\frac{q^2}{m^2} = 1$ is expressed by the Dirac equation in energy-momentum space,

$$\left.\begin{array}{l} s\left(\frac{q}{m}\right)\circ\hat{s}^{-1}\left(\frac{q}{m}\right)=\frac{\sigma^j q_j}{m}\Rightarrow s\left(\frac{q}{m}\right)=\frac{\sigma^j q_j}{m}\circ\hat{s}\left(\frac{q}{m}\right)\\ \hat{s}\left(\frac{q}{m}\right)\circ s^{-1}\left(\frac{q}{m}\right)=\frac{\check{\sigma}^j q_j}{m}\Rightarrow \hat{s}\left(\frac{q}{m}\right)=\frac{\check{\sigma}^j q_j}{m}\circ s\left(\frac{q}{m}\right)\end{array}\right\}\Rightarrow\left(\gamma^j q_j - m\right)\mathbf{s}\left(\frac{q}{m}\right)=0$$

$$\text{with } \gamma^j=\begin{pmatrix}0 & \sigma^j\\ \check{\sigma}^j & 0\end{pmatrix},\quad \mathbf{s}\left(\frac{q}{m}\right)=\begin{pmatrix}s\left(\frac{q}{m}\right) & 0\\ 0 & \hat{s}\left(\frac{q}{m}\right)\end{pmatrix}.$$

The four columns of the (4×4) matrix $\mathbf{s}(\frac{q}{m})$ are familiar as solutions of the Dirac equation.

For the Pauli transmutator, the analogue to the Dirac equation is the condition $\sigma^a x_a u(\frac{\vec{x}}{r}) - u(\frac{\vec{x}}{r})\sigma_3 r = 0$, which restricts the three parameters to two independent perpendicular ones.

An action from left with the Lorentz group $\mathbf{SL}(\mathbb{C}^2)$ gives the Weyl transmutator at the Lorentz transformed energy-momenta $\Lambda.q$ on the hyperboloid $q^2=m^2$, accompanied by a right action with a Wigner spin $\mathbf{SU}(2)$-rotation

$$\lambda\in\mathbf{SL}(\mathbb{C}^2):\ \lambda\circ s\left(\frac{q}{m}\right)=s\left(\frac{\Lambda.q}{m}\right)\circ u\left(\frac{q}{m},\lambda\right)$$

$$\text{with}\quad\begin{cases} u\left(\frac{q}{m},\lambda\right)\in\mathbf{SU}(2),\\ \Lambda.q=\lambda\circ q\circ\lambda^\star,\\ \Lambda_j^k=\frac{1}{2}\operatorname{tr}\sigma_j\circ\lambda\circ\check{\sigma}^k\circ\lambda^\star\in\mathbf{SO}_0(1,3).\end{cases}$$

3.4. Higgs Transmutators

In the standard model of electroweak interactions the three weak parameters for the Goldstone manifold of electromagnetic relativity are implemented by three chargelike degrees of freedom of the Higgs vector $\Phi^\alpha\cong\binom{\Phi^1}{\Phi^2}\in V\cong\mathbb{C}^2$ and its orthogonal $\epsilon^{\alpha\beta}\Phi^\star_\beta$,

$$\mathbb{C}^2\supset\mathcal{G}^3\ni\frac{\Phi}{M}\longmapsto v\left(\frac{\Phi}{M}\right)\in\mathbf{U}(2),$$

$$\begin{aligned} v\left(\frac{\Phi}{M}\right) &= \begin{pmatrix} e^{i(\alpha_3-\alpha_0)}\cos\frac{\theta}{2} & -e^{-i(\varphi-\alpha_0)}\sin\frac{\theta}{2}\\ e^{i(\varphi-\alpha_0)}\sin\frac{\theta}{2} & e^{-i(\alpha_3-\alpha_0)}\cos\frac{\theta}{2}\end{pmatrix}\\ &= u(\varphi-\alpha_3,\theta)\circ e^{i(\alpha_3-\alpha_0)\sigma_3}\\ &= \frac{1}{M}\begin{pmatrix}\Phi_2^\star & \Phi^1\\ -\Phi_1^\star & \Phi^2\end{pmatrix}\text{ with }\det v\left(\frac{\Phi}{M}\right)=\frac{|\Phi^1|^2+|\Phi^2|^2}{M^2}=1.\end{aligned}$$

The restriction from four to three real weak degrees of freedom uses the $\mathbf{U}(2)$-invariant scalar product $\langle\Phi|\Phi\rangle = M^2$ of the Higgs vector space.

A left hypercharge-isospin action on the fundamental Higgs transmutator gives the transmutator at the $\mathbf{U}(2)$-transformed Higgs vector on the Goldstone manifold, accompanied by a Wigner electromagnetic $\mathbf{U}(1)_+$-transformation from right

$$u \in \mathbf{U}(2) : u \circ v\left(\frac{\Phi}{M}\right) = v\left(\frac{u.\Phi}{M}\right) \circ u_+ \text{ with } \begin{cases} u = e^{i\gamma_0} u_2 \in \mathbf{U}(1) \circ \mathbf{SU}(2), \\ u_+ = \begin{pmatrix} e^{i2\gamma_0} & 0 \\ 0 & 1 \end{pmatrix} \in \mathbf{U}(1)_+. \end{cases}$$

3.5. Real Tetrads (Vierbeins)

For general relativity, the 10-parametric $\mathbf{GL}(\mathbb{R}^4)$-orbit of the orthonormal $\mathbf{O}(1,3)$-'idolized" Lorentz metric in a symmetric matrix $\eta = \eta^T$ is diagonalizable to four principal axes with a transformation from a maximal compact subgroup $\mathbf{O}(4) \subset \mathbf{GL}(\mathbb{R}^4)$ (6 parameters),

$$\gamma = h \circ \eta \circ h^T = \gamma^T = O(\gamma) \circ \operatorname{diag}\gamma \circ O(\gamma)^T \text{ with } O(\gamma) \in \mathbf{O}(4).$$

The diagonal part of the metric hyperboloid, multiplied by the inverse metric η^{-1}, displays the remaining four dilation transformations from the maximal noncompact abelian subgroup,

$$\eta^{-1} \circ \operatorname{diag}\gamma = e^{2[d(\gamma)+d_0(\gamma)]} \in \mathbf{D}(1) \times \mathbf{SO}_0(1,1)^3 \cong \mathbf{D}(1)^4 \subset \mathbf{GL}(\mathbb{R}^4).$$

The diagonal elements are four directional units, one for time and three for the metric ellipsoid of 3-position.

The operational decomposition of the metric hyperboloid leads to the parametrization of the 10-dimensional tetrad h as basis of real 4-dimensional tangent spacetime $\mathbb{R}^4$ with four dilations and a 6-dimensional rotation

$$\mathcal{M}^{10} \ni \gamma \longmapsto h(\gamma) \in \mathbf{D}(1)^4 \times \mathbf{O}(4) \subset \mathbf{GL}(\mathbb{R}^4), \quad h(\gamma) = e^{d(\gamma)+d_0(\gamma)} \circ O(\gamma).$$

A general linear $\mathbf{GL}(\mathbb{R}^4)$ left-multiplication gives the tetrad for a transformed metric tensor and a Wigner right-transformation by the idolized Lorentz group $\mathbf{O}(1,3)$,

$$g \in \mathbf{GL}(\mathbb{R}^4) : g \circ h(\gamma) = h(g \circ \gamma \circ g^T) \circ \Lambda(g,\gamma) \text{ with } \Lambda(g,\gamma) \in \mathbf{O}(1,3).$$

3.6. Complex Dyads (Zweibeins)

Nonlinear spacetime $\mathcal{D}^4$, i.e., the orientation manifold of $\mathbf{U}(2)$-scalar products for unitary relativity, is parametrizable by the future cone,

$$x \in \mathcal{D}^4 \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) = \mathbf{D}(\mathbf{1}_2) \times \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathbf{D}(1) \times \mathcal{Y}^3.$$

It is transformed to an "idolized" diagonal scalar product by a Weyl transmutator $s(\frac{x}{\sqrt{x^2}}) \in \mathbf{SL}(\mathbb{C}^2)$ for the three hyperbolic degrees of freedom and a dilation $\mathbf{D}(1) = \exp \mathbb{R} \cong \mathbb{R}$ for eigentime $e^{2\beta_0} = \sqrt{x^2}$,

$$\begin{aligned} x &= \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix} = s\left(\frac{x}{\sqrt{x^2}}\right) \circ e^{2\beta_0 \mathbf{1}_2} \circ s^\star\left(\frac{x}{\sqrt{x^2}}\right) \\ &= u\left(\frac{\vec{x}}{r}\right) \circ \operatorname{diag} x \circ u^\star\left(\frac{\vec{x}}{r}\right), \\ \operatorname{diag} x &= \begin{pmatrix} x_0 + r & 0 \\ 0 & x_0 - r \end{pmatrix} = e^{2(\beta_0 \mathbf{1}_2 + \beta\sigma_3)} \in \mathbf{D}(1) \times \mathbf{SO}_0(1,1) \\ &\text{with } e^{4\beta_0} = x^2, \ \tanh 2\beta = \frac{r}{x_0}. \end{aligned}$$

The diagonalization of the scalar products gives the fundamental transmutator from the extended Lorentz group to the hyperisospin subgroup. It is a basis of the complex 2-dimensional space and will be called, in analogy to a real tetrad or vierbein, a complex dyad or zweibein. It is parametrized by the future cone spacetime points as orbit of the $\mathbf{U}(2)$-scalar product,

$$\begin{aligned} \mathbb{R}^4 \supset \mathcal{D}^4 \ni x &\longmapsto \psi(x) \in \mathbf{GL}(\mathbb{C}^2), \\ \psi(x) \circ \mathbf{1}_2 \circ \psi^\star(x) &= x, \\ \psi(x) &= s\left(\frac{x}{\sqrt{x^2}}\right) \circ e^{\beta_0 \mathbf{1}_2} = u\left(\frac{\vec{x}}{r}\right) \circ e^{\beta_0 \mathbf{1}_2 + \beta\sigma_3}. \end{aligned}$$

The left action with the extended Lorentz group $\mathbf{GL}(\mathbb{C}^2)$ as external transformation gives the dyad ψ at a Lorentz transformed and dilated spacetime point in the future cone, accompanied by an action from right with an internal spacetime dependent Wigner hyperisospin $\mathbf{U}(2)$-transformation,

$$g \in \mathbf{GL}(\mathbb{C}^2): \ g \circ \psi(x) = \psi(e^{2\delta_0} \Lambda.x) \circ u(x, g)$$

$$\text{with} \quad \begin{cases} u(x,g) \in \mathbf{U}(2), \\ g = e^{\delta_0 + i\alpha_0} \lambda \in \mathbf{D}(1) \times \mathbf{U}(1) \circ \mathbf{SL}(\mathbb{C}^2), \\ g \circ x \circ g^\star = e^{2\delta_0} \lambda \circ x \circ \lambda^\star = e^{2\delta_0} \Lambda.x, \quad \Lambda \in \mathbf{SO}_0(1,3). \end{cases}$$

4. LINEAR REPRESENTATIONS OF RELATIVITIES

In the foregoing section, the classes G/H of the five relativities with linear groups were represented by defining linear transformations. The products of these fundamental transmutators give the finite-dimensional representations of the homogeneous spaces G/H.

4.1. Rectangular Transmutators

Representations of the "general" group G involve representations of the cosets G/H representatives,

$$G \ni g \longmapsto D(g) \in \mathbf{GL}(V),$$
$$G/H \ni gH \ni g_r \longmapsto D(g_r),$$

e.g. for perpendicular relativity

$$\mathbf{SU}(2) \ni u(\varphi, \theta, \chi) = \begin{pmatrix} e^{i\frac{\chi+\varphi}{2}} \cos\frac{\theta}{2} & -e^{i\frac{\chi-\varphi}{2}} \sin\frac{\theta}{2} \\ e^{-i\frac{\chi-\varphi}{2}} \sin\frac{\theta}{2} & e^{-i\frac{\chi+\varphi}{2}} \cos\frac{\theta}{2} \end{pmatrix}$$
$$\longmapsto u(\varphi, \theta, \chi)_r = u(\varphi, \theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\varphi} \sin\frac{\theta}{2} \\ e^{i\varphi} \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in \mathbf{SU}(2)/\mathbf{SO}(2) \cong \Omega^2$$

A G-representation can be decomposed into H-representations. In an $(n \times n)$ matrix representation,

$$D(g) \in V \otimes V^T \cong \mathbb{C}^n \otimes \mathbb{C}^n \overset{\text{e.g.,}}{\sim} \begin{pmatrix} \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \\ \bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet&\bullet \end{pmatrix}$$

with an (8×8)-example for $\mathbf{SU}(3)$ and the octet decomposition $8 = 2 + 1 + 3 + 2$ into $\mathbf{SU}(2)$-representations,

$$V \overset{H}{\cong} \bigoplus_{\iota=1}^{k} V_\iota, \quad H \bullet V_\iota \subseteq V_\iota, \quad D(h) \overset{H}{\cong} \bigoplus_{\iota=1}^{k} d_\iota(h) \overset{\text{e.g.,}}{\sim} \left(\begin{array}{cc|c|ccc|cc} \bullet&\bullet&0&0&0&0&0&0 \\ \bullet&\bullet&0&0&0&0&0&0 \\ \hline 0&0&\bullet&0&0&0&0&0 \\ \hline 0&0&0&\bullet&\bullet&\bullet&0&0 \\ 0&0&0&\bullet&\bullet&\bullet&0&0 \\ 0&0&0&\bullet&\bullet&\bullet&0&0 \\ \hline 0&0&0&0&0&0&\bullet&\bullet \\ 0&0&0&0&0&0&\bullet&\bullet \end{array}\right),$$

the G-representation matrices can be decomposed into rectangular $(n \times n_\iota)$ matrices, $n_\iota \leq n$,

$$D(g) = \bigoplus_{\iota=1}^{k} D_\iota(g) = (D_1(g)|D_2(g)|\cdots|D_I(g)) \overset{\text{e.g.,}}{\sim} \left(\begin{array}{cc|c|cccc|cc} \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array}\right)$$

with left-right $G \times H$-action, e.g., the $\mathbf{SU}(3) \times \mathbf{SU}(2)$-action on octet-dublet, octet-singlet, octet-triplet and octet-dublet. As mappings of the coset representatives $(G/H)_r$, they are called transmutators:

$$\begin{gathered} (G/H)_r \ni g_r \longmapsto D_\iota(g_r) \in V \otimes V_\iota^T \cong \mathbb{C}^n \otimes \mathbb{C}^{n_\iota} \\ \text{with} \begin{cases} D_\iota(g_r h) = D_\iota(g_r) \circ d_\iota(h), \; h \in H, \\ D_\iota(k g_r) = D(k) \circ D_\iota(g_r) \\ \qquad\qquad = D((kg)_r) \circ d_\iota(h(k, g_r)), \; k \in G. \end{cases} \end{gathered}$$

With bases of the G-vector spaces $|D; j\rangle \in V$ and the H-vector spaces $|\iota; a\rangle \in V_\iota$ one has in a Dirac notation with kets for vectors $|\ \rangle \in V$ and bras for linear forms $\langle\ | \in V_\iota^T$

$$\begin{aligned} V \otimes V_\iota^T \ni D_\iota(g_r) &= |D; j\rangle D_\iota(g_r)^j_a \langle \iota; a|, \\ \text{e.g. } \mathbb{C}^8 \otimes \mathbb{C}^2 \ni D_2(g_r) &= |8; j\rangle D_2(g_r)^j_a \langle 2; a|, \; j = 1 \ldots, 8; \; a = 1, 2. \end{aligned}$$

The finite-dimensional transmutators are $(n \times n_\iota)$-dimensional vector spaces with $G \times H$-representations. Those representations are unitary, called Hilbert representations, only for the compact relativities, i.e., in the examples above, for perpendicular and electromagnetic relativity. There, the transmutators are complete for the harmonic analysis of the Hilbert spaces with the square integrable functions $L^2(G/H)$ of the orientation manifold of the relativity (more below).

4.2. Representations of Perpendicular Relativity

For perpendicular relativity, all transmutators from rotations to axial rotations arise by the totally symmetric products, denoted by $\overset{2J}{\bigvee}$, of the fundamental Pauli transmutator $u(\frac{\vec{x}}{r}) \in \mathbf{SU}(2)$,

$$\mathbf{SU}(2)/\mathbf{SO}(2) \cong \Omega^2 \longrightarrow \mathbf{SU}(1+2J,)$$
$$\frac{\vec{x}}{r} \longmapsto [2J]\left(\frac{\vec{x}}{r}\right) = \bigvee^{2J} u\left(\frac{\vec{x}}{r}\right), \ \vec{x}^2 = r^2.$$

The irreducible spin $\mathbf{SU}(2)$-representations $[2J]$ are decomposable into axial rotation $\mathbf{SO}(2)$-representations (n) with dimension 2 for $n \neq 0$ and two polarizations $\pm n$ (left- and right-circulary polarized) :

$$\textbf{irrep}\,\mathbf{SU}(2) \ni [2J] \overset{\mathbf{SO}(2)}{\cong} \begin{cases} \bigoplus_{n=0,2,..}^{2J} (n) \text{ for } J = 0, 1, \ldots, \\ \bigoplus_{n=1,3,..}^{2J} (n) \text{ for } J = \frac{1}{2}, \frac{3}{2}, \ldots, \end{cases}$$

e.g., for rotations acting on 3-position $\mathbb{R}^3$ with $a, b \in \{1, 2, 3\}$ and $\alpha, \beta \in \{1, 2\}$:

$$[2]\left(\frac{\vec{x}}{r}\right) \cong O\left(\frac{\vec{x}}{r}\right)_a^b = \frac{1}{2}\operatorname{tr} u\left(\frac{\vec{x}}{r}\right) \circ \sigma^b \circ u^\star\left(\frac{\vec{x}}{r}\right) \circ \sigma^a$$
$$= \frac{1}{r}\left(\begin{array}{c|c} \delta^{\alpha\beta} r - \frac{x_\alpha x_\beta}{r+x_3} & x_\alpha \\ -x_\beta & x_3 \end{array}\right) \in \mathbf{SO}(3),$$
$$[2] \overset{\mathbf{SO}(2)}{\cong} (2) \oplus (0)$$

with the relations for the $\mathbf{SO}(3)$ and $\mathbf{SO}(2)$ metric tensors

$$O\left(\frac{\vec{x}}{r}\right)_{\alpha,3}^a \delta_{ab} O\left(\frac{\vec{x}}{r}\right)_{\beta,3}^b = \left(\begin{array}{c|c} \delta_{\alpha\beta} & 0 \\ \hline 0 & 1 \end{array}\right), \quad O\left(\frac{\vec{x}}{r}\right)_\alpha^a \delta^{\alpha\beta} O\left(\frac{\vec{x}}{r}\right)_\beta^b = \delta^{ab} - \frac{x_a x_b}{r^2}.$$

The 2nd symmetric power of the Pauli transmutator, in a Cartesian and a spherical basis,

$$O\left(\frac{\vec{x}}{r}\right) = \left(\begin{array}{c|c} \delta^{\alpha\beta} - \frac{x_\alpha x_\beta}{r(r+x_3)} & \frac{x_\alpha}{r} \\ -\frac{x_\beta}{r} & \frac{x_3}{r} \end{array}\right) \cong \left(\begin{array}{cc|c} e^{i\varphi}\cos^2\frac{\theta}{2} & -e^{i\varphi}\sin^2\frac{\theta}{2} & ie^{i\varphi}\frac{\sin\theta}{\sqrt{2}} \\ i\frac{\sin\theta}{\sqrt{2}} & i\frac{\sin\theta}{\sqrt{2}} & \cos\theta \\ -e^{-i\varphi}\sin^2\frac{\theta}{2} & e^{-i\varphi}\cos^2\frac{\theta}{2} & ie^{-i\varphi}\frac{\sin\theta}{\sqrt{2}} \end{array}\right),$$

displays in the 3rd column $O(\frac{\vec{x}}{r})_3^b$ the spherical harmonics $\mathrm{Y}_1(\varphi, \theta) \sim \frac{\vec{x}}{r}$ as a basis for the $\mathbb{C}^3$-Hilbert subspace in the Hilbert space $L^2(\Omega^2)$ with the square integrable functions on the 2-sphere. Its symmetric traceless products of power $J = 1, 2, \ldots$ give the spherical harmonics $\mathrm{Y}_J(\varphi, \theta) \sim (\frac{\vec{x}}{r})_{\text{traceless}}^J$ which arise as the $(1+2J)$-entries in one column of the $(1+2J) \times (1+2J)$ matrices for the representation

$[2J]$. The spherical harmonics are bases for the Hilbert spaces $\mathbb{C}^{1+2J} \subset L^2(\Omega^2)$ with the irreducible $\mathbf{SO}(3)$-representations.

With respect to the dichotomic $\mathbf{SU}(2) \times \mathbf{SO}(2)$-transformation behavior, the four functions in the two columns of the (2×2)-Pauli transmutator,

$$u\left(\frac{\vec{x}}{r}\right) = \begin{pmatrix} \sqrt{\frac{x_3+r}{2r}} & -\frac{x_1-ix_2}{\sqrt{2r(x_3+r)}} \\ \frac{x_1+ix_2}{\sqrt{2r(x_3+r)}} & \sqrt{\frac{x_3+r}{2r}} \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \in \mathbb{C}^2 \otimes \mathbb{C}^2,$$

and the six functions in the 1st and 2nd column $O(\frac{\vec{x}}{r})^b_{1,2} \cong O(\frac{\vec{x}}{r})^b_{+,-}$ above in a rectangular (3×2) matrix constitute bases for finite-dimensional Hilbert spaces $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^3 \otimes \mathbb{C}^2$ with $\mathbf{SU}(2)$-representations on $\mathbb{C}^2$ and $\mathbb{C}^3$, acting from left, and nontrivial $\mathbf{SO}(2)$-representations $\mathbf{SO}(2) \ni e^{i\alpha_3\sigma_3} \longmapsto e^{in\alpha_3\sigma_3}$ on $\mathbb{C}^2$ with $n = 1, 2$ respectively, acting from right. They are irreducible subspaces in the harmonic analysis of the Hilbert space $L^2(\Omega^2, \mathbb{C}^2)$ with the square integrable mappings from the 2-sphere into a vector space with nontrivial $\mathbf{SO}(2)$-action.

In general one has the Peter-Weyl decompositions Peter and Weyl (1927) into irreducible subspaces for $\mathbf{SU}(2) \times \mathbf{SO}(2)$ action:

$$V_n \cong \mathbb{C}^{2-\delta_{n0}}: \quad L^2(\Omega^2, V_n) \cong \bigoplus_{2Jn} \mathbb{C}^{1+2J} \otimes V_n \text{ (dense).}$$

The orthogonal sum goes over all $\mathbf{SU}(2)$-representation that contain the $\mathbf{SO}(2)$-representation on $V_n \cong \mathbb{C}, \mathbb{C}^2$. This generalizes the case for the spherical harmonics with $V_0 \cong \mathbb{C}$.

4.3. Representations of Rotation Relativity

For special relativity, all finite-dimensional 3-hyperboloid representations (boost representations), i.e., all finite-dimensional transmutators from Lorentz group to rotation group, can be built by the totally symmetric products of the two fundamental Weyl transmutators $s(\frac{q}{m}), \hat{s}(\frac{q}{m}) \in \mathbf{SL}(\mathbb{C}^2)$,

$$\mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \cong \mathcal{Y}^3 \longrightarrow \mathbf{SL}(\mathbb{C}^{(1+2L)(1+2R)}),$$

$$\frac{q}{m} \longmapsto [2L|2R]\left(\frac{q}{m}\right) = \bigvee^{2L} s\left(\frac{q}{m}\right) \otimes \bigvee^{2R} \hat{s}\left(\frac{q}{m}\right), \quad q^2 = m^2.$$

The finite-dimensional irreducible Lorentz group representations can be decomposed into irreducible spin representations,

$$\mathbf{irrep}^{\,\text{finite}}\mathbf{SL}(\mathbb{C}^2) \ni [2L|2R] \overset{\mathbf{SO}(2)}{\cong} \bigoplus_{J=|L-R|}^{L+R} [2J].$$

For example, the vector representation $\Lambda = [1|1]$ gives two irreducible transmutators from Lorentz group to rotation group, the first column for spin 0-representation and the three remaining columns for spin 1-representation, with $a, b \in \{1, 2, 3\}$,

$$[1|1]\left(\frac{q}{m}\right) = \Lambda\left(\frac{q}{m}\right)^j_k \cong \frac{1}{2}\operatorname{tr} s\left(\frac{q}{m}\right) \circ \sigma^j \circ s^\star\left(\frac{q}{m}\right) \circ \check{\sigma}_k$$
$$= \frac{1}{m}\left(\begin{array}{c|c} q_0 & q_a \\ q_b & \delta_{ab} m + \frac{q_a q_b}{m+q_0} \end{array}\right) \in \mathbf{SO}_0(1,3),$$
$$[1|1] \overset{\mathbf{SO}(2)}{\cong} [0] \oplus [2].$$

The four columns of the matrix $\Lambda(\frac{q}{m})^j_{0,a}$ relate to each other the metric tensors of $\mathbf{SO}_0(1, 3)$ and $\mathbf{SO}(3)$

$$\Lambda\left(\frac{q}{m}\right)^k_{0,a} \eta_{kj} \Lambda\left(\frac{q}{m}\right)^j_{0,b} = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & -\delta_{ab} \end{array}\right), \quad \Lambda\left(\frac{q}{m}\right)^k_a \delta^{ab} \Lambda\left(\frac{q}{m}\right)^j_b = -\eta^{kj} + \frac{q^k q^j}{m^2}.$$

The transmutators from Lorentz group to rotation group in the rectangular (4×3)-submatrix $\Lambda(\frac{q}{m})^k_a \in \mathbb{R}^4 \otimes \mathbb{R}^3$ are used for massive spin 1 particles in Lorentz vector fields, e.g., for the neutral weak boson and its Feynman propagator,

$$\mathbf{Z}(x)^j = \oint \frac{d^3q}{2q_0} \Lambda\left(\frac{q}{m}\right)^j_a [e^{iqx} \mathrm{u}(\vec{q})^a + e^{-iqx} \mathrm{u}^\star(\vec{q})^a],$$
$$\langle\{\mathbf{Z}^k(y), \mathbf{Z}^j(x)\} - \epsilon(x_0 - y_0)[\mathbf{Z}^k(y), \mathbf{Z}^j(x)]\rangle$$
$$= \frac{i}{\pi} \int \frac{d^4q}{(2\pi)^3} \frac{\left(-\eta^{kj} + \frac{q^k q^j}{m^2}\right)}{q^2 + io - m^2} e^{iq(x-y)}.$$

In contrast to compact perpendicular relativity with the Hilbert space $L^2(\Omega^2)$, the Hilbert space for the square integrable functions on the 3-hyperboloid $L^2(\mathcal{Y}^3)$ has no finite-dimensional Hilbert subspaces with irreducible $\mathbf{SL}(\mathbb{C}^2)$-representations. The monomials in the columns of the fundamental Weyl transmutators give bases for finite-dimensional $\mathbf{SL}(\mathbb{C}^2) \times \mathbf{SU}(2)$ representations on $\mathbb{C}^{(1+2L)(1+2R)} \otimes \mathbb{C}^{1+2J}$, which are indefinite unitary for the noncompact Lorentz group $\mathbf{SL}(\mathbb{C}^2)$. The spin $\mathbf{SU}(2)$-representation has to be contained in the $\mathbf{SL}(\mathbb{C}^2)$-representation, i.e., $|L - R| \leq J \leq L + R$.

4.4. Representations of Electromagnetic Relativity

For the standard model of electroweak interactions, the Higgs parametrized defining representation of the orientation manifold $\mathcal{G}^3$ of the electromagnetic group with hypercharge $y = \frac{1}{2}$ and isospin $T = \frac{1}{2}$ and its conjugate, i.e., the two Higgs transmutators

$$\left[\frac{1}{2}|1\right]\left(\frac{\Phi}{M}\right) = v\left(\frac{\Phi}{M}\right) = \frac{1}{M}\left(\begin{matrix}\Phi_2^\star \\ -\Phi_1^\star\end{matrix}\middle|\begin{matrix}\Phi^1 \\ \Phi^2\end{matrix}\right), \quad |\Phi|^2 = M^2,$$

$$\left[-\frac{1}{2}|1\right]\left(\frac{\Phi}{M}\right) = v^\star\left(\frac{\Phi}{M}\right),$$

give, via their products, all irreducible representations of the Goldstone manifold:

$$\mathbf{U}(2)/\mathbf{U}(1)_+ \cong \mathcal{G}^3 \longrightarrow \mathbf{U}(1+2T), \qquad \frac{\Phi}{M} \longmapsto [\pm n + T|2T]\left(\frac{\Phi}{M}\right),$$

$$\mathbf{irrep}\,\mathbf{U}(2) \ni [\pm n + T|2T] \cong [\pm 1|0]^n \otimes \bigvee^{2T}\left[\frac{1}{2}|1\right],$$

$$\mathbf{irrep}\,\mathbf{U}(1) \ni [\pm 1|0] \cong \left[\pm\frac{1}{2}|1\right] \wedge \left[\pm\frac{1}{2}|1\right].$$

Because of the central correlation $\mathbf{SU}(2) \cap \mathbf{U}(\mathbf{1}_2) = \{\pm\mathbf{1}_2\}$ in $\mathbf{U}(2)$, the $\mathbf{U}(2)$-representations have the correlation of the hypercharge- and isospin-invariant $y = T \pm n$ with natural n, i.e., the two invariants (y, T) for the rank 2 $\mathbf{U}(2)$-transformations are either both integer or both halfinteger as visible in the colorless fields of the standard model.

The decomposition of a hyperisospin $\mathbf{U}(2)$-representation into irreducible representations of the electromagnetic group $\mathbf{U}(1)_+ \ni e^{i2\gamma_0} \longmapsto e^{zi2\gamma_0}$ is given with integer charge numbers $z \in \mathbb{Z}$:

$$\mathbf{U}(2) \quad \ni [\pm n + T|2T] \overset{\mathbf{U}(1)_+}{\cong} \bigoplus_{z=\pm n}^{\pm n+2T} [z],$$

$$\text{e.g.,} \quad \begin{cases} [\pm\frac{1}{2}|1] \cong [0] \,\oplus\, [\pm 1], \\ \quad [0|2] \cong [-1] \,\oplus\, [0] \,\oplus\, [1]. \end{cases}$$

The (1×1) examples with antisymmetric power 2 of the fundamental Higgs transmutators give the transmutation from hyperisospin $\mathbf{U}(2)$ to electromagnetic $\mathbf{U}(1)_+$ on $\mathbb{C}$ for hypercharge nontrivial isospin $\mathbf{SU}(2)$-singlets with charge numbers $z = \pm 1$,

$$[1|0]\left(\frac{\Phi}{M}\right) = \frac{\tilde{\Phi}_\alpha^\star \Phi^\alpha}{M^2} \in \mathbf{U}(1) \text{ with } [1|0] \overset{\mathbf{U}(1)_+}{\cong} [1], \quad \tilde{\Phi}^\alpha = \epsilon^{\alpha\beta}\Phi_\beta^\star,$$

$$[-1|0]\left(\frac{\Phi}{M}\right) = \frac{\Phi_\alpha^\star \tilde{\Phi}^\alpha}{M^2} \in \mathbf{U}(1) \text{ with } [-1|0] \overset{\mathbf{U}(1)_+}{\cong} [-1].$$

The (3×1) product of both fundamental Higgs transmutators describes an hypercharge trivial isospin $\mathbf{SU}(2)$-triplet. The columns are three transmutators to charge $z \in \{-1, 0, 1\}$,

$$[0|2]\left(\frac{\Phi}{M}\right) = \frac{1}{2} \operatorname{tr} \tau^b \circ v\left(\frac{\Phi}{M}\right) \circ \tau^a \circ v^\star\left(\frac{\Phi}{M}\right)$$
$$= \left(\frac{\Phi^\star\vec{\tau}\tilde{\Phi}+\tilde{\Phi}^\star\vec{\tau}\Phi}{2M^2} \middle| \frac{\Phi^\star\vec{\tau}\tilde{\Phi}-\tilde{\Phi}^\star\vec{\tau}\Phi}{2iM^2} \middle| \frac{\tilde{\Phi}^\star\vec{\tau}\tilde{\Phi}-\Phi^\star\vec{\tau}\Phi}{2M^2} \right) \in \mathbf{SO}(3)$$
$$\text{with } [0|2] \overset{\mathbf{U}(1)_+}{\cong} [-1] \oplus [0] \oplus [1].$$

These three transmutators are used for the transition from the three isospin gauge fields in the electroweak standard model to the weak boson particles

$$\tau^a \mathbf{A}_a(x) = \mathbf{A}(x) \longmapsto v\left(\frac{\Phi(x)}{M}\right) \circ \mathbf{A}(x) \circ v^\star\left(\frac{\Phi(x)}{M}\right)$$
$$+ \left[\partial v\left(\frac{\Phi(x)}{M}\right)\right] \circ v^\star\left(\frac{\Phi(x)}{M}\right)$$
$$= (\mathbf{W}_-(x), \mathbf{W}_0(x), \mathbf{W}_+(x)) = (\mathbf{A}_-(x), \mathbf{A}_0(x), \mathbf{A}_+(x)) + \dots$$
$$\text{with } [0|2] \quad \left(\frac{\Phi(x)}{M}\right)_i^a = \delta_i^a + \dots$$

For the definition of particles with the transition from Lorentz group to rotation group the neutral field $\mathbf{W}_0$ is combined, in the Weinberg rotation, with the hypercharge gauge field.

Similar to perpendicular relativity, the Hilbert spaces of the square integrable mappings of the compact Goldstone manifold $L^2(\mathcal{G}^3, V_z)$ into a Hilbert space $V_z \cong \mathbb{C}$ with electromagnetic action $\mathbf{U}(1)_+ \ni e^{i2\gamma_0} \longmapsto e^{zi2\gamma_0}$ have Peter-Weyl decompositions into finite-dimensional subspaces $\mathbb{C}^{1+2T} \otimes \mathbb{C}$ with irreducible representations of $\mathbf{U}(2) \times \mathbf{U}(1)$ where the isospin representations fulfill $2T \geq |z|$. The representation spaces are given by the columns in the products of the fundamental Higgs transmutator and its conjugate.

4.5. Representations of Unitary Relativity

All finite-dimensional representations of unitary relativity, i.e., of nonlinear spacetime $\mathcal{D}^4$,

$$\mathcal{D}^4 \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2) \cong \mathbf{D}(1) \times \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \longrightarrow \mathbf{GL}(\mathbb{C}^{(1+2L)(1+2R)}),$$

use products of the two conjugated dyads, e.g.,

$$\begin{aligned}
\psi(x) &= u\left(\frac{\vec{x}}{r}\right) \circ e^{\beta_0 \mathbf{1}_2 + \beta\sigma_3} \in \mathbf{GL}(\mathbb{C}^2),\\
\psi(x) \circ \mathbf{1}_2 \circ \psi^\star(x) &= u\left(\frac{\vec{x}}{r}\right) \circ e^{2(\beta_0 \mathbf{1}_2 + \beta\sigma_3)} \circ u^\star\left(\frac{\vec{x}}{r}\right) = x \in \mathbf{GL}(\mathbb{C}^2),\\
(\psi(x) \circ \mathbf{1}_2 \circ \psi^\star(x))^2 &= e^{4\beta_0} = x^2 \in \mathbf{D}(1).
\end{aligned}$$

The monomials in the dyads span finite-dimensional spaces with $\mathbf{GL}(\mathbb{C}^2) \times \mathbf{U}(2)$-representations that are indefinite unitary for the noncompact group $\mathbf{GL}(\mathbb{C}^2)$. They are no Hilbert spaces and, therefore, of little importance for a quantum structure. Hilbert spaces with faithful representations of nonlinear spacetime have to be infinite dimensional. They will be treated in "Relativities and homogeneous spaces II –Spacetime as unitary relativity."

4.6. Representations of Lorentz Group Relativity

For general relativity, all finite-dimensional representations of the general linear group $\mathbf{GL}(\mathbb{R}^4)$ for the tetrads,

$$\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1,3) \cong \mathbf{D}(1) \times \mathbf{SL}_0(\mathbb{R}^4)/\mathbf{SO}_0(1,3), \quad \mathbf{SL}_0(\mathbb{R}^4) \sim \mathbf{SO}_0(3,3),$$

are obtained by products of the fundamental representations of the rank 3 special subgroup $\mathbf{SL}_0(\mathbb{R}^4)$, which is locally isomorphic to the indefinite orthogonal group $\mathbf{SO}_0(3,3)$. The three fundamental representations are the two 4-dimensional spinor representations, dual to each other, and a 6-dimensional self-dual one,

$$\dim_{\mathbb{R}}[1,0,0] = 4, \quad \dim_{\mathbb{R}}[0,1,0] = \binom{4}{2} = 6, \quad \dim_{\mathbb{R}}[0,0,1] = \binom{4}{3} = 4.$$

The dimensions of the finite-dimensional irreducible $\mathbf{SL}_0(\mathbb{R}^4)$-representations are given by the Weyl formula:

$$\begin{aligned}
&\mathbf{irrep}^{\,\text{finite}}\mathbf{SL}_0(\mathbb{R}^4) \ni [n_1, n_2, n_3] \cong \mathbb{N}^3,\\
&d_n = \dim_{\mathbb{R}}[n_1, n_2, n_3]\\
&\quad = \frac{(n_1+1)(n_2+1)(n_3+1)(n_1+n_2+2)(n_3+n_2+2)(n_1+n_2+n_3+3)}{12},\\
&\quad \text{dual reflection: } [n_1, n_2, n_3] \leftrightarrow [n_3, n_2, n_1].
\end{aligned}$$

Self-dual representation spaces, i.e., for $[n, m, n]$, have an $\mathbf{SL}(\mathbb{R}^4)$-invariant symmetric bilinear form with neutral signature.

The finite-dimensional representations of Lorentz group relativity, parametrized by the metric manifold,

$$\mathbf{GL}(\mathbb{R}^4)/\mathbf{O}(1,3) \cong \mathcal{M}^{10} \ni \gamma \longmapsto [n_1, n_2, n_3](\gamma) \in \mathbf{GL}(\mathbb{R}^{d_n}),$$

have decompositions with respect to an "idolized" Lorentz group. The three fundamental $\mathbf{SL}_0(\mathbb{R}^4)$-representations give the two fundamental $\mathbf{SO}_0(1,3)$-representations, i.e., the 4-dimensional Minkowski representation and the 6-dimensional adjoint representation,

$$\begin{aligned} [1,0,0], [0,0,1] &\overset{\mathbf{SO}_0(1,3)}{\cong} [1|1], \\ [0,1,0] &\overset{\mathbf{SO}_0(1,3)}{\cong} [2|0] \oplus [0|2]. \end{aligned}$$

The totally antisymmetric powers of the defining $\mathbf{SL}_0(\mathbb{R}^4)$-representation combine the two other fundamental ones

$$\bigwedge^2 [1,0,0] = [0,1,0], \quad \bigwedge^3 [1,0,0] = [0,0,1], \quad \bigwedge^4 [1,0,0] = [0,0,0]$$

They can be realized by the tetrad, a spinor representation, as fundamental transmutator from general linear group to Lorentz group and its totally antisymmetric powers

$$\begin{aligned} h_j^\mu(\gamma) \in \mathbf{GL}(\mathbb{R}^4), \quad h_{\kappa\lambda}^{lm}(\gamma) &= \epsilon_{\mu\nu\kappa\lambda}\epsilon^{jklm} h_j^\mu(\gamma) h_k^\nu(\gamma) \in \mathbf{GL}(\mathbb{R}^6), \\ h_\lambda^m(\gamma) &= \epsilon_{\mu\nu\kappa\lambda}\epsilon^{jklm} h_j^\mu(\gamma) h_k^\nu(\gamma) h_k^\kappa(\gamma) \in \mathbf{GL}(\mathbb{R}^4) \\ \det h(\gamma) &\in \mathbf{GL}(\mathbb{R}) \end{aligned}$$

The tetrad power-2 product is a (6×6) transmutator acted on by the self-dual fundamental $\mathbf{SL}_0(\mathbb{R}^4)$-representation and the adjoint Lorentz group representation. It has the same transformation properties as the curvature tensor

$$\gamma \longmapsto R_{\kappa\lambda}^{lm}(\gamma) \in \mathbb{R}^6 \otimes \mathbb{R}^6 \quad \text{with} \quad \begin{array}{c|c} & [2|0] \oplus [0|2] \\ \hline [0,1,0] & R_{\kappa\lambda}^{lm}(\gamma) \end{array}$$

The determinant with power 4 is an $\mathbf{SL}(\mathbb{R}^4)$-scalar with nontrivial $\mathbf{D}(1)$-dilation properties.

Obviously, all those real finite-dimensional representation spaces of the noncompact product $\mathbf{GL}(\mathbb{R}^4) \times \mathbf{SO}_0(1,3)$ have no invariant Hilbert product. A harmonic analysis of, e.g., square integrable functions $L^2(\mathcal{M}^{10})$ on the metric manifold $\mathcal{M}^{10} \cong \mathbf{GL}(\mathbb{R}^4)/\mathbf{SO}_0(1,3)$, does not play a role in classical gravity.

5. RELATIVITY REPRESENTATIONS BY INDUCTION

Finite-dimensional rectangular mappings of homogeneous spaces G/H (H-relativity), as discussed in the foregoing section, give all Hilbert representation spaces only for compact relativities, e.g., for perpendicular and electromagnetic relativity. In general, the faithful $G \times H$-Hilbert representations of a locally compact relativity G/H, infinite for noncompact G, can be induced by representations of the "idolized" subgroup H.

5.1. Induced Representations

Induced representations are $G \times H$-subrepresentations of the two-sided regular $G \times G$-representation. They are the extension of the left G-action $gH \overset{_kL}{\longmapsto} kgH$ on the right H-cosets in the form of linear transformations.

The vector spaces for subgroup H-induced G-representations consist of H-intertwiners on the group $w : G \longrightarrow W$ with values in a Hilbert space with a unitary action of the "idolized" subgroup $d : H \longrightarrow \mathbf{U}(W)$. The G-action on the intertwiners is defined by left multiplication $_kL$, all this is expressed in the commutative diagram:

$$\begin{array}{ccc} G & \overset{_kL \times R_h}{\longrightarrow} & G \\ w \downarrow & & \downarrow {_kw}, \\ W & \underset{d(h)}{\longrightarrow} & W \end{array} \qquad \begin{array}{rl} g, k \in G,\ h \in H : & {_kL} \times R_h(g) = kgh^{-1}, \\ H\text{-intertwiner:} & w(gh^{-1}) = d(h).w(g), \\ G\text{-action:} & w \longmapsto {_kw}, \\ & {_kw}(g) = w(k^{-1}g). \end{array}$$

An H-intertwiner on the group $w \in W^{G/H}$ maps H-cosets of the group into H-orbits in the Hilbert space W. It is defined by its values on representatives $g_r \in (G/H)_r \subseteq G$. The G-action comes with the representative dependent H-action ("gauge group action") of the related Wigner element $h(g_r, k) \in H$,

$$\begin{array}{ccc} (G/H)_r & \overset{_kL}{\longrightarrow} & (G/H)_r \\ w \downarrow & & \downarrow {_kw}, \\ W & \longrightarrow & W \end{array} \qquad \begin{array}{l} G \times W^{(G/H)_r} \longrightarrow W^{(G/H)_r}, \\ {_{k^{-1}}w}(g_r) = w(kg_r) = d(h^{-1}(g_r, k)).w((kg)_r). \end{array}$$

The induced representation may be reducible. Since the fixgroups for the left G-action on the right H-cosets are conjugates of H,

$$G_{gH} = \{k \in G \,\big|\, kgH = gH\} = gHg^{-1} \cong H,$$

each G-representation on $W^{(G/H)_r}$ and its subspaces contains the inducing H-representation d.

With a G-left invariant coset measure $dg_r = dkg_r$, the intertwiners, in a bra vector notation, have a direct integral expansion with the cosets as natural distributive basis $\langle g_r, a|$ and complex coefficients $w(g_r)_a \in \mathbb{C}$

$$\langle w| = {}^{\oplus}\!\!\int_{(G/H)_r} dg_r\ w(g_r)_a \langle g_r, a| \in W^{(G/H)_r}.$$

The G-invariant Hilbert product integrates the Hilbert product of the value space W over the cosets

$$W^{(G/H)_r} \times W^{(G/H)_r} \longrightarrow \mathbb{C}, \quad \| w \|^2 = \int_{(G/H)_r} dg_r\ \overline{w(g_r)_a} w(g_r)_a,$$

An orthonormal distributive basis is defined with a Dirac distribution $\delta(g_r, g_r')$, supported by the relativity manifold and normalized with respect to the invariant measure used dg_r (examples below)

$$\begin{aligned} &\langle g_r', a' | g_r, a\rangle = \delta_{aa'} \delta(g_r, g_r') \\ &\text{with } \langle w | g_r, a\rangle = \int_{(G/H)_r} dg_r'\ \delta(g_r, g_r') w(g_r')_a = w(g_r)_a. \end{aligned}$$

In the simplest case, the functions on the homogeneous G-space for H-relativity, valued in the complex numbers as 1-dimensional space $W = \mathbb{C}\langle 1|$ with trivial H-action $d_0(h) = 1$, are expanded as direct integral over the cosets with the corresponding function values

$$\langle f| : (G/H)_r \longrightarrow \mathbb{C}, \ \ \langle f| = {}^{\oplus}\!\!\int_{(G/H)_r} dg_r\ f(g_r)\langle g_r|.$$

They are matrix elements (coefficients) of G-representations D which contain a trivial H-representation $D \supseteq d_0$.

5.2. Transmutators as Induced Representations

The transmutators above, valued in finite-dimensional rectangular matrices, are acted on with $G \times H$-representations, a G-action from left, induced by an

H-action from right

$$\begin{array}{ccc} & (G/H)_r & \xrightarrow{{}_kL} & (G/H)_r & \\ D_\iota & \downarrow & & \downarrow & {}_kD_\iota, \\ & V_D \otimes V_\iota^T & \longrightarrow & V_D \otimes V_\iota^T & \end{array} \quad {}_{k^{-1}}D_\iota(g_r) = D_\iota(kg_r) = D(k) \circ D_\iota(g_r) \\ = D((kg)_r) \circ d_\iota(h(k, g_r)),$$

Transmutators can be used for a decomposition of any G-representation induced by a H-representation d_ι on a vector space with basis $\langle \iota; a| \in V_\iota^T$,

$$w_\iota : (G/H)_r \longmapsto V_\iota^T, \quad \langle w_\iota| = {}^{\oplus}\!\!\int_{(G/H)_r} dg_r \; w_\iota(g_r)_a \langle \iota; g_r, a|.$$

There occur all G-representations D, which contain the inducing H-representation d_ι. With a basis $|D; j\rangle \in V_D$ one obtains the harmonic D-components $\tilde{w}_\iota(D)_j$, which come with multiplicity n_D,

$$\langle w_\iota^{\text{finite}}| = \bigoplus_{D \supseteq d_\iota} n_D \tilde{w}_\iota(D)_j \; \langle D_\iota^j| \text{ with } \begin{cases} \langle D_\iota^j| = {}^{\oplus}\!\!\int_{(G/H)_r} dg_r \; D_\iota(g_r)_a^j \langle \iota; g_r, a|, \\ \tilde{w}_\iota(D)_j = \langle w_\iota | D; j\rangle, \end{cases}$$

$$w_\iota^{\text{finite}}(g_r)_a = \bigoplus_{D \supseteq d_\iota} n_D \tilde{w}_\iota(D)_j D(g_r)_a^j, \; w_\iota^{\text{finite}}(kg_r)_a = \bigoplus_{D \supseteq d_\iota} n_D \tilde{w}_\iota(D)_j D(k)_k^j D(g_r)_a^k,$$

e.g., the harmonic analysis of functions with the harmonic D-components $\tilde{f}(D)_j$,

$$\langle f^{\text{finite}}| = \bigoplus_{D \supseteq d_0} n_D \tilde{f}(D)_j \langle D_0^j| \text{ with } \begin{cases} \langle D_0^j| = {}^{\oplus}\!\!\int_{(G/H)_r} dg_r \; D(g_r)_0^j \langle g_r|, \\ \tilde{f}(D)_j = \langle f | D; j\rangle. \end{cases}$$

5.3. The Hilbert Spaces of Compact Relativities

For a compact "general" group G, the finite-dimensional rectangular transmutators are square integrable on the manifolds G/H. They are complete for the harmonic analysis of the group G and its homogeneous spaces G/H, i.e., they exhaust by orthogonal direct Peter-Weyl decompositions with Schur orthogonality all square integrable induced representations,

$$\text{compact } G : \begin{cases} \langle w_\iota| = \langle w_\iota^{\text{finite}}|, \\ L^2(G/H, V_\iota^T) \cong \bigoplus_{D \supseteq d_\iota} n_D \; V_D \otimes V_\iota^T \text{ (dense)}, \\ \text{id}_{(V_\iota^T)^{G/H}} = \bigoplus_{D \supseteq d_\iota} n_D \; \text{id}_{V_D} \cong \bigoplus_{D \supseteq d_\iota} n_D \; |D; j\rangle\langle D; j|. \end{cases}$$

There is Frobenius' reciprocity theorem for the number n_D of d_ι-induced G-representations D.

As discussed above, all representation matrix elements of the compact groups $\mathbf{U}(2)$ and $\mathbf{SU}(2)$ are square integrable with the finitely decomposable Hilbert spaces for electromagnetic relativity $L^2(\mathcal{G}^3, V_z)$ and perpendicular relativity $L^2(\Omega^2, V_{|z|})$.

The complex functions for perpendicular relativity are the spherical harmonics[4] as products of the three matrix elements $\vec{\omega} \longmapsto \sqrt{\frac{4\pi}{3}}\mathrm{Y}_1(\vec{\omega})^a \in \mathbb{C}$ in the middle column with trivial representations of $\mathbf{SO}(2) \ni e^{i\chi}$ and a triplet representation of $\mathbf{SO}(3)$,

$$\begin{pmatrix} e^{i(\chi+\varphi)}\cos^2\frac{\theta}{2} & ie^{i\varphi}\frac{\sin\theta}{\sqrt{2}} & -e^{-i(\chi-\varphi)}\sin^2\frac{\theta}{2} \\ ie^{i\chi}\frac{\sin\theta}{\sqrt{2}} & \cos\theta & ie^{-i\chi}\frac{\sin\theta}{\sqrt{2}} \\ -e^{i(\chi-\varphi)}\sin^2\frac{\theta}{2} & ie^{-i\varphi}\frac{\sin\theta}{\sqrt{2}} & e^{-i(\chi+\varphi)}\cos^2\frac{\theta}{2} \end{pmatrix} \in \mathbf{SO}(3),$$

$$\mathbf{SU}(2)/\mathbf{SO}(2) \cong \Omega^2 \ni \frac{\vec{x}}{r} = \vec{\omega} \longmapsto [2J](\vec{\omega})^a_0 = \sqrt{\frac{4\pi}{1+2J}}\mathrm{Y}_J(\vec{\omega})^a \in \mathbb{C}$$

$$\text{for } J = 0, 1, 2, \ldots \text{ with } a \in \{-J, \ldots, +J\},$$

$$O \in \mathbf{SO}(3) : [2J](O)^a_b \mathrm{Y}_J(\vec{\omega})^b = \mathrm{Y}_J(O.\vec{\omega})^a.$$

There is Schur's orthogonality Schur (1905); Folland (1995); Knapp (1986) with the Plancherel normalization given by the dimension $1 + 2J$ of the representation space,

$$\int_{\Omega^2} \frac{d^2\omega}{4\pi}\, \overline{[2J](\vec{\omega})^a_0}\, [2J'](\vec{\omega})^{a'}_0 = \int_{\Omega^2} d^2\omega\, \frac{\overline{\mathrm{Y}_J(\vec{\omega})^a}}{\sqrt{1+2J}}\, \frac{\mathrm{Y}_{J'}(\vec{\omega})^{a'}}{\sqrt{1+2J'}} = \frac{1}{1+2J}\delta_{JJ'}\delta^{aa'}.$$

It involves the rotation invariant normalizable measure and the distributive basis of the 2-sphere,

$$\int_{\Omega^2} d^2\omega = \int_0^{2\pi} d\varphi \int_{-1}^{1} d\cos\theta = 4\pi,$$

$$\langle\vec{\omega}'|\vec{\omega}\rangle = \delta(\vec{\omega} - \vec{\omega}') = \delta(\varphi - \varphi')\frac{1}{\sin\theta}\delta(\theta - \theta').$$

[4] In the Euler angle parametrization, both the middle column and the middle row define the $\mathbf{SO}(3)$-action on the 2-sphere $\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2) \cong \mathbf{SO}(2)\backslash\mathbf{SO}(3)$. The central element $\theta \longmapsto \cos\theta$ parametrizes the double coset space, the 1-sphere $\Omega^1 \cong \mathbf{SO}(2)\backslash\mathbf{SO}(3)/\mathbf{SO}(2) \cong \mathbf{SO}(2)$ and is a spherical Ω^2-function.

The spherical harmonics exhaust the square integrable 2-sphere functions,

$$L^2(\Omega^2) \ni \langle f| = {}^{\oplus}\!\!\int_{\Omega^2} d^2\omega f(\vec{\omega})\langle\vec{\omega}| = \bigoplus_{J=0}^{\infty} \tilde{f}(J)_a {}^{\oplus}\!\!\int_{\Omega^2} d^2\omega \, \mathrm{Y}_J(\vec{\omega})^a \langle\vec{\omega}|,$$

$$f(\vec{\omega}) = \bigoplus_{J=0}^{\infty} \tilde{f}(J)_a \;\; \mathrm{Y}_J(\vec{\omega})^a.$$

The measure can be rewritten with a 2-sphere-supported Dirac distribution

$$r\langle[2J]_0^a| \sim {}^{\oplus}\!\!\int_{\Omega^2} d^2\omega \, \mathrm{Y}_J(\vec{\omega})\langle\vec{\omega}| = {}^{\oplus}\!\!\int d^3x \; \delta(\vec{x}^2 - 1)(\vec{x})^J_{\text{traceless}}\langle\vec{x}|,$$

$$\text{with } \left(\frac{\vec{x}}{|\vec{x}|}\right)^J_{\text{traceless}} = [2J](\vec{\omega}).$$

5.4. Finite-dimensional Analysis of Special Relativity

The harmonic analysis of free quantum fields with respect to the eigenvectors for spacetime translations and spin rotations uses non-Hilbert representations of the Lorentz group.

Induced representations of a noncompact group also contain finite-dimensional rectangular transmutators. For example, the three matrix elements in the middle column with trivial representations of $\mathbf{SO}(2) \ni e^{i\chi}$,

$$\left(\begin{array}{c|c|c} e^{i(\chi+\varphi)}\cosh^2\beta & e^{i\varphi}\frac{\sinh 2\beta}{\sqrt{2}} & -e^{-i(\chi-\varphi)}\sinh^2\beta \\ e^{i\chi}\frac{\sinh 2\beta}{\sqrt{2}} & \cosh 2\beta & e^{-i\chi}\frac{\sinh 2\beta}{\sqrt{2}} \\ -e^{i(\chi-\varphi)}\sinh^2\beta & e^{-i\varphi}\frac{\sinh 2\beta}{\sqrt{2}} & e^{-i(\chi+\varphi)}\cosh^2\beta \end{array}\right) \in \mathbf{SO}_0(1,2),$$

are complex functions on the 2-hyperboloid $\mathbf{SO}_0(1,2)/\mathbf{SO}(2) \cong \mathcal{Y}^2 \longrightarrow \mathbb{C}$ with a triplet representation of $\mathbf{SO}_0(1,2)$. However, there are no finite-dimensional faithful Hilbert representations of noncompact Lie groups.

Relativistic quantum fields for massive particles involve rectangular transmutators acted on with $\mathbf{SL}(\mathbb{C}^2) \times \mathbf{SU}(2)$-representations. For example, the representation $\Lambda = [1|1]$ of the Lorentz group in the $\mathbf{SO}_0(1,3) \times \mathbf{SO}(3)$-representation on $\mathbb{C}^4 \otimes V_\iota^T$ for a special relativistic vector field and the tensor representation $\Lambda \wedge \Lambda = [2|0] \oplus [0|2]$ on $\mathbb{C}^6 \otimes V_\iota^T$ for its field strength,

$$\mathbf{Z}(0)^j = {}^{\oplus}\!\!\int_{\mathcal{Y}^3} \frac{d^3q}{2q_0} \Lambda\left(\frac{q}{m}\right)^j_a [\mathrm{u}(\vec{q})^a + \mathrm{u}^\star(\vec{q})^a],$$

$$i\mathbf{F}(0)^{kj} = {}^{\oplus}\!\!\int_{\mathcal{Y}^3} \frac{d^3q}{2q_0} \Lambda\left(\frac{q}{m}\right)^l_0 \epsilon^{kj}_{lr} \Lambda\left(\frac{q}{m}\right)^r_a [\mathrm{u}(\vec{q})^a - \mathrm{u}^\star(\vec{q})^a]$$

$$\text{with } q_0 = \sqrt{m^2 + \vec{q}^2}, \;\; \epsilon^{kj}_{lr} = \delta^k_l\delta^j_r - \delta^k_r\delta^j_l, \; a = 1,2,3, \;\; j = 0,1,2,3,$$

are both induced by an $\mathbf{SO}(3)$-representation (Folland, 1995) on $V_\iota \cong \mathbb{C}^3$ and its dual V_ι^T. These spin representations act on the creation and annihilation operators $\mathrm{u}(\vec{q})^a, \mathrm{u}^\star(\vec{q})^a$ for a massive particle with momentum $\vec{q}$ and spin 1 directions $a = 1, 2, 3$. The action of the creation operators on the Fock ground state $|0\rangle$ gives dual distributive bases of the special relativistic manifold, i.e., of the energy-momentum hyperboloid $\mathcal{Y}^3 \cong \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$ for mass m^2. The distributive orthogonality is given by the Fock expectation value $\langle\ \ \rangle$,

$$|1; \vec{q}, a\rangle = \mathrm{u}(\vec{q})^a|0\rangle \in V_\iota(\vec{q}), \quad \langle 1; \vec{q}, a| = \langle 0|\mathrm{u}^\star(\vec{q})^a \in V_\iota^T(\vec{q}),$$
$$\langle 1; \vec{p}, b|1; \vec{q}, a\rangle = \langle \mathrm{u}^\star(\vec{p})^b \mathrm{u}(\vec{q})^a\rangle = \delta^{ab} 2\sqrt{m^2 + \vec{q}^2}\delta(\vec{q} - \vec{p}).$$

The 4- and 6-dimensional Lorentz group representations do not act on Hilbert spaces. That can be seen at the transmutators from Lorentz group to rotation group, e.g., $\{\frac{\vec{q}}{m} \longmapsto \Lambda(\frac{q}{m})^j_a\}$, which are not square integrable $L^2(\mathcal{Y}^3)$ on the energy-momentum hyperboloid.

The Lorentz invariant nonnormalizable measure of the 3-hyberboloid in the momentum parametrization cabn be written as integral with a $\mathcal{Y}^3$-supported Dirac distribution,

$$\int_{\mathcal{Y}^3} \frac{d^3q}{2\sqrt{m^2 + \vec{q}^2}} = \int d^4q\ \vartheta(q_0)\delta(q^2 - m^2).$$

The finite-dimensional Lorentz group $\mathbf{SL}(\mathbb{C}^2)$-representations that contain a trivial rotation group $\mathbf{SU}(2)$-representation are $[n|n]$. They act on vector spaces $\mathbb{C}^{(1+n)(1+n)}$, $n = 0, 1, \ldots$, with the spin-representation decomposition:

$$\mathbf{irrep}\,^{\text{finite}}\mathbf{SO}_0(1, 3) \ni [n|n] \overset{\mathbf{SO}(3)}{\cong} \bigoplus_{J=0}^{n} [2J] \text{ e.g., } \begin{cases} [0|0] \cong [0], \\ [1|1] \cong [0] \oplus [2], \\ [2|2] \cong [0] \oplus [2] \oplus [4]. \end{cases}$$

They are used for finite-dimensional $\mathbf{SO}_0(1, 3)$-representation expansion of complex functions on energy-momentum hyperboloids

$$\langle [n|n]_0^{j_1\ldots j_n}| : \quad \mathcal{Y}^3 \longrightarrow \mathbb{C} \text{ for } n = 0, 1, 2, \ldots$$
$$\langle [n|n]_0^{j_1\ldots j_n}| = {}^{\oplus}\!\!\int_{\mathcal{Y}^3} \frac{d^3q}{2q_0} [n|n]_0^{j_1\ldots j_n}(\vec{q})\langle \vec{q}| \text{ with } q_0 = \sqrt{m^2 + \vec{q}^2}$$
$$= {}^{\oplus}\!\!\int d^4q\ \vartheta(q_0)\delta(q^2 - m^2)[n|n]_0^{j_1\ldots j_n}(q)\langle q|$$

and arise as contributions of the Feynman propagators for trivial translations, e.g., for a spin 0 particle in a scalar field $\mathbf{\Phi}$, a spin $\frac{1}{2}$-particle in a Dirac field $\mathbf{\Psi}$, and a spin 1 particle in a vector field **Z**:

$$\mathbf{\Phi}(0): \quad \langle[0|0]| =^{\oplus}\!\!\int d^4q\ \vartheta(q_0)\delta(q^2-m^2)\langle q|,$$

$$\mathbf{\Psi}(0): \langle[0|0]| \oplus \gamma_j\langle[1|1]_0^j| =^{\oplus}\!\!\int d^4q\ \vartheta(q_0)\delta(q^2-m^2)\left(\mathbf{1}_4+\frac{\gamma_j q^j}{m}\right)\langle q|,$$

$$\mathbf{Z}^j(0): \quad \langle[2|2]_0^{jk}| =^{\oplus}\!\!\int d^4q\ \vartheta(q_0)\delta(q^2-m^2)\left(-\eta^{kj}+\frac{q^kq^j}{m^2}\right)\langle q|.$$

The spacetime translation dependent fields, e.g., a massive vector field,

$$\mathbb{R}^4 \ni x \longmapsto \mathbf{Z}(x)^j, \mathbf{F}(x)^{kj} = \epsilon^{kj}_{lr}\frac{\partial^l}{m}\mathbf{Z}(x)^r$$

involve $e^{iqx}\mathrm{u}(\vec{q})^a$ and $e^{-iqx}\mathrm{u}^\star(\vec{q})^a$, which are the translation orbits $\mathbb{R}^4 \ni x \longmapsto e^{\pm iqx} \in \mathbf{U}(1)$ for a representation of the Poincaré group $\mathbf{SO}_0(1,3)\ \vec{\times}\ \mathbb{R}^4$. This leads to the spacetime translation representation coefficients with $\langle q|x\rangle = e^{iqx}$ and $\langle x|q\rangle = e^{-iqx}$ as the on-shell part of the Feynman propagator,

$$\begin{aligned}\langle[2|2]_0^{jk}|x\rangle+\left\langle x[2|2]_0^{jk}\right\rangle &= \int d^4q\,\delta(q^2-m^2)\left(-\eta^{kj}+\frac{q^kq^j}{m^2}\right)e^{iqx}\\ &= \langle\{\mathbf{Z}^k(y),\mathbf{Z}^j(x)\}\rangle.\end{aligned}$$

REFERENCES

Bacon, F. (1994). *Novum Organon* (1620), P. Urban and J. Gibson (tr. eds), Open Court Publishing Company, Peru, Illinois.

Folland, G. B. (1995). *A Course in Abstract Harmonic Analysis*, CRC Press, Boca Raton, Ann Arbor, London, Tokyo.

Frobenius, F. G. (1968). *Gesammelte Abhandlungen*, Springer Verlag, Berlin.

Helgason, S. (1984). *Groups and Geometric Analysis*, Academic Press, New York. London, Sydney, Tokyo, Toronto, etc.

Knapp, A. (1986). *Representation Theory of Semisimple Groups*, Princeton University Press, Princeton.

Mackey, G. W. (1951). *Amer. J. Math.* **73**, 576–592.

Peter, F. and Weyl, H. (1927). *Math. Ann.* **97**, 737–755.

Saller, H. (2001). *International Journal of Theoretical Physics* **40**, 1151–1172 hep-th/0011265.

Schur, I. (1905). *Sitzungsber. Preuss. Akad.*, 406.

Treves, F. (1967). *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York, London.

Weinberg, S. (1967). *Physics and Review Letter* **18**, 507.

Weyl, H. (1923). *Raum, Zeit, Materie* Wissenschaftliche Buchgesellschaft, Darmstadt.

Wigner, E. P. (1939). *Annals of Mathematics* **40**, 149–204.